## Year 11 Term 4 Mathematics Extension 1, 2008

**Question 1** Marks

(a) Solve for x **:** $\dfrac{3x+2}{2x} \geq 4$ **3**

(b) Find the general solution to **:** $1 + 2\sin x = 0$ **2**

(c) Differentiate with respect to $x$ and simplify **:**

(i) $2\tan^{-1}\left(\dfrac{3x}{4}\right)$ **3**

(ii) $\cos^{-1}(2x-3)+\sin^{-1}(2x-3)$ **1**

**Question 2**

(a) Evaluate $\displaystyle\int_1^2 x\sqrt{x^2+2}\,dx$ using the substitution $u = x^2+2$. **3**

(b) Find $\displaystyle\int \frac{4dx}{\sqrt{25-4x^2}}$ **2**

(c) (i) Express $2\sqrt{3}\sin x - 2\cos x$ in the form $R\cos(x+\alpha)$ where $R>0$ and $-\dfrac{\pi}{2} \leq x \leq \pi$. **2**

(ii) Hence or otherwise solve **:** $2\sqrt{3}\sin x - 2\cos x > 2$ for $-\dfrac{\pi}{2} \leq x \leq \pi$. **2**

**Question 3**

(a) (i) Find $f'(x)$ if $f(x) = \tan^{-1}x + \tan^{-1}\left(\dfrac{1}{x}\right)$. **2**

(ii) Deduce an equivalent function for $f(x)$. **1**

(iii) Hence graph the function $y = \tan^{-1}x + \tan^{-1}\left(\dfrac{1}{x}\right)$. **2**

(b) (i) Graph the curve $y = 3\cos^{-1}x$. **2**

(ii) Find the exact area of the region bounded by the curve $y = 3\cos^{-1}x$, the lines $y = \pi$ and the $y$ axis. **2**

**Question 4**

(a) (i) On the same axes shade the region defined by **:** **3**

$y \le 2x+1$ , $y \ge x^2 - 2$ and $x \ge 2$.

(ii) Show that the point $P(3,7)$ is an intersection point of $y = 2x+1$ and $y = x^2 - 2$. **2**

(iii) Find the volume of revolution when the region defined in (a)(i)is rotated about the $x$ axis. **4**

**Question 5**

(a) If $A = \cos^{-1}\left(\dfrac{8}{17}\right)$ find the value of *tan2A.* **2**

(b) Evaluate $\displaystyle\int_2^{2\sqrt{3}} \frac{x-2}{\sqrt{16-x^2}}\,dx$ **4**

(c) If $f(x) = 2^x + 5$ find the inverse function $f^{-1}(x)$, stating the domain and range of $f^{-1}(x)$. **3**

**Question 6**

(a) Evaluate to 3 decimal places $\displaystyle\int_0^4 \log_{10}\left(x^2+3\right)dx$ using Simpson's Rule with 3 function values. **3**

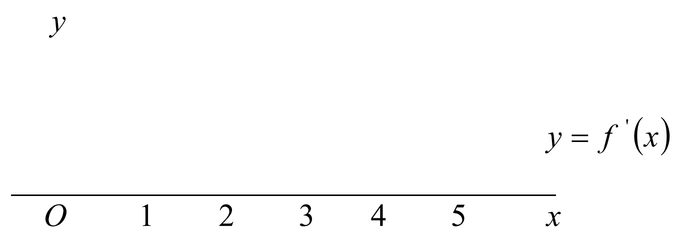
(b) (i) Prove $\dfrac{x+y}{2} \ge \sqrt{xy}$ for all $x \ge 0$ and $y \ge 0$. **2**

(ii) Hence prove that the area of a rectangle cannot exceed the average of the square of the lengths of it's two sides. **1**

(iii) If $a > 0$, $b > 0$, $c > 0$, and $a + b + c = 1$, show that $(1-a)(1-b)(1-c) \ge 8abc$ . **3**

**Question 7**

(a) The graph below shows the gradient function $f'(x)$. **2**




Find the values of $x$ where $y = f(x)$ has a local minimum. Justify your answer.

(b) A truck is to travel 1000 kilometres at a constant speed of $v$ km/h. When travelling at $v$ km/hr, the truck consumes fuel at the rate of

$$\left(6+\frac{v^2}{50}\right) \text{ litres per hour.}$$

The truck company pays 50 cents/litre for fuel and pays each of the two drivers $20 per hour whilst the truck is travelling.

**2**

(i) Let the total cost of fuel and the drivers' wages for the trip be $C$ dollars. Show that **:**

$$C = 10v + 43000\frac{1}{v}.$$

(ii) The truck must take no longer than 12 hours to complete the trip, and speed limits require that $v \le 100$ . **5**

At what speed $v$ should the truck travel to minimize the cost $C$ ?

**End of Exam**

Q1 SU Y4 T4.

# YR11 T4 2008 Ext 1

(a) $\frac{3x+2}{2x} \geq 4 \quad x \neq 0.$

$2x(3x+2) \geq 16x^2$

$x\left[3x+2-8x\right] \geq 0.$

$x\left[2-5x\right] \geq 0$

$0 < x \leq \frac{2}{5}$ (3)

(b) $\sin x = -\frac{1}{2}$

$x = \pi k + (-1)^k \sin^{-1}\left(-\frac{1}{2}\right)$

$= \pi k - (-1)^k \cdot \frac{\pi}{6}$ $k$ integer. (2)

(c)(i) $\frac{d}{dx}\, 2\tan^{-1}\left(\frac{3x}{4}\right) = 2 \cdot \frac{3}{4} \cdot \frac{1}{1+\left(\frac{3x}{4}\right)^2}$

$= \frac{3}{2} \cdot \frac{1}{1+\frac{9x^2}{16}}$

$= \frac{24}{16+9x^2}$ (3)

(ii) $\frac{d}{dx}\left[\cos^{-1}(2x-3) + \sin^{-1}(2x-3)\right] = \frac{d}{dx}\left(\frac{\pi}{2}\right)$

$= 0.$ (1)

Q2(a) $\int_1^2 x\sqrt{x^2+2}\,dx$ $\quad u = x^2+2 \quad x=1 \;\; u=3$

$du = 2x\,dx \quad x=2 \;\; u=6$

$= \int_3^6 u^{\frac{1}{2}} \frac{du}{2}$

$= \left[\frac{u^{\frac{3}{2}}}{3}\right]_3^6$

$= 2\sqrt{6} - \sqrt{3}$ (3)

(b) $\int \frac{4\,dx}{\sqrt{25-4x^2}} = \frac{4}{2}\int \frac{dx}{\sqrt{\frac{25}{4} - x^2}}$

$= 2\sin^{-1}\left(\frac{x}{\frac{5}{2}}\right) + C$

$= 2\sin^{-1}\left(\frac{2x}{5}\right) + C$ (2).

c(i) $2\sqrt{3}\sin x - 2\cos x = R\cos(x+\alpha)$ $\quad R \geq 0.$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\sin\alpha = -2\sqrt{3}$ $\quad \sin\alpha > 0$ $\quad \pi < \alpha < \frac{3\pi}{2}$

$R\cos\alpha = -2$ $\quad \cos\alpha > 0$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{4\pi}{3}$

$R = \sqrt{(2\sqrt{3})^2 + 2^2}$

$= 4$

$2\sqrt{3}\sin x - 2\cos x = 4\cos\left(x + \frac{4\pi}{3}\right)$

(ii)

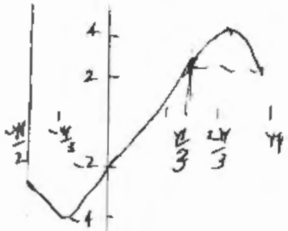
$4\cos\left(x+\frac{4\pi}{3}\right)=2$

$\cos\left(x+\frac{4\pi}{3}\right)=\frac{1}{2}$

$x+\frac{4\pi}{3}=2\pi k\pm\frac{\pi}{3}$

$x=\pi$ or $\frac{\pi}{3}$

$\therefore\ \frac{\pi}{3}<x<2\pi$




(2)

Qu3 (i) $f(x)=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)$

$f'x=\frac{1}{1+x^2}+\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{-1}{x^2}$

$=\frac{1}{1+x^2}\quad\frac{-1}{x^2\left(1+\frac{1}{x^2}\right)}$

$=\frac{1}{1+x^2}\quad\frac{-1}{1+x^2}$

$=0$

(2).

(ii) $\therefore f(x)=$ constant

$x>0\quad x=1\quad f(x)=\tan^{-1}1+\tan^{-1}1$

$=\frac{\pi}{4}+\frac{\pi}{4}$

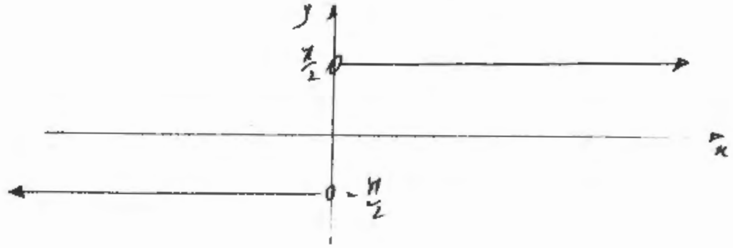
$=\frac{\pi}{2}$ ($\frac{1}{2}$)

$x<0\quad x=-1$

$f(x)=-\frac{\pi}{4}-\frac{\pi}{4}$

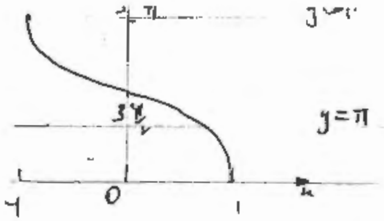
$=-\frac{\pi}{2}$ ($\frac{1}{2}$)

iii)




(2)

3b (i) $y=3\cos^{-1}x$




(2)

(ii) $x=\cos\frac{y}{3}$

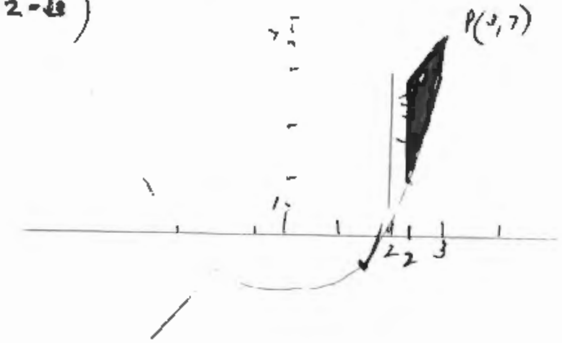
Area $=\int_{\pi}^{\frac{3\pi}{2}}\cos\left(\frac{y}{3}\right)dy$

$=\left[3\sin\frac{y}{3}\right]_{\pi}^{\frac{3\pi}{2}}$

$=3\left[1-\frac{\sqrt{3}}{2}\right]$

$=\frac{3}{2}\left(2-\sqrt{3}\right)$ (2)

Qu4 (i)




(3).

(ii) $y=2x+1$ $\qquad$ $y=x^2-2$

$x=3\quad y=7$ $\qquad$ $x=3\quad y=3^2-2$

$y=7$

$P(3,7)$ satisfies both $y=2x+1$ & $y=x^2-2$

$\therefore$ $P(3,7)$ is an intersection point. (2)

(iii) Volume $V=\pi\int_{2}^{3}(2x+1)^2-(x^2-2)^2\,dx$ (1)

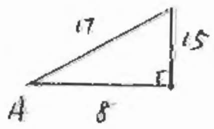
$=\pi\left\{\left[\frac{(2x+1)^3}{6}\right]_2^3-\int_2^3 x^4-4x^2+4\,dx\right\}$ (1)

$V = \pi \left\{ \frac{7^3 - 5^3}{6} - \left( \frac{x^5}{5} - \frac{4x^3}{3} + 4x \right)_2^3 \right\}$ (1)

$= \pi \left\{ \frac{109}{3} - \left( \frac{243}{5} - 36 + 12 - \left( \frac{32}{5} - \frac{32}{3} + 8 \right) \right) \right\}$

$= \frac{232\pi}{15}$ unit³ (1)

Qu 5



$0 < A < \frac{\pi}{2}$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

$= \frac{2 \cdot \frac{15}{8}}{1 - \frac{225}{64}}$

$= \frac{240}{64 - 225}$

$= -\frac{240}{161}$ (2)

(b) $\int_2^{2\sqrt{3}} \frac{x-2}{\sqrt{16-x^2}}\,dx = \int_2^{2\sqrt{3}} \left( \frac{x}{\sqrt{16-x^2}} - \frac{2}{\sqrt{16-x^2}} \right) dx$

$= \left[ -\sqrt{16-x^2} - 2\sin^{-1}\frac{x}{4} \right]_2^{2\sqrt{3}}$ (2)

$= -2 - 2 \cdot \frac{\pi}{3} - \left( -\sqrt{12} - 2 \cdot \frac{\pi}{6} \right)$

$= 2\sqrt{3} - 2 - \frac{\pi}{3}$ (2)

(c) $x = 2^y + 5$ $\quad x > 5$

$2^y = x - 5$

$y = \log_2 (x-5)$

$f^{-1}x = \log_2 (x-5)$ (1)

Domain $f^{-1}x$ $\quad x > 5$ (1)

Range $f^{-1}x$ defined all values y (1)

Qu 6

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | $\log_{10}3$ | $\log_{10}7$ | $\log_{10}19$ |

a) $\therefore \int_0^4 \log_{10}(x^2+3)\,dx \approx \frac{2}{3}\left[ \log_{10}3 + 4\log_{10}7 + \log_{10}19 \right]$

$\approx \frac{2}{3} \log_{10} 136857$

$\approx 3.424$ (3)

b)(i) $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$

$\therefore x - 2\sqrt{x}\sqrt{y} + y \geq 0$

$x + y \geq 2\sqrt{xy}$.

$\frac{x+y}{2} \geq \sqrt{xy}$. (2)

(ii) $x = l^2 \quad y = b^2$

$\frac{l^2 + b^2}{2} \geq \sqrt{l^2 b^2}$

$l^2 + b^2 \geq 2lb$.

$\geq 2A$

$\therefore A \leq \frac{l^2 + b^2}{2}$ (1)

iii) $(1-a)(1-b)(1-c)$

$= (b+c)(a+c)(a+b)$ ①

$\geq 2\sqrt{bc}\ 2\sqrt{ac}\ 2\sqrt{ab}$ ①

$\geq 8\sqrt{a^2b^2c^2}$ ①

$\geq 8abc$. ③

Qu 7 (a) For minimum $f'x = 0$

$f''x > 0$ (slope of $f'x$ at $x$). ②

$\Rightarrow x = 5$

(b)(i) Cost = Cost Drivers + Cost fuel $\qquad t = \frac{1000}{v}$ hrs

$= \$2 \times 20 \times \frac{1000}{v} + \frac{1000}{v}\left(6 + \frac{v^2}{50}\right) \times \$0.50$. ①

$= \$\left(\frac{40000}{v} + \left(\frac{6000}{v} + 20v\right) \times 0.50\right)$

$= \$\left(\frac{40000}{v} + \frac{3000}{v} + 10v\right)$ ①

$C = \$\left(\frac{43000}{v} + 10v\right)$

(ii) For minimum cost $\frac{dC}{dv} = -\frac{43000}{v^2} + 10$ ①

$\frac{d^2C}{dv^2} = \frac{86000}{v^3}$

For minimum cost $\frac{dC}{dv} = 0$.

$-\frac{43000}{v^2} + 10 = 0$

$v^2 = 4300$

$v = 65.57$ km/hr. ①

Time journey = 15.25 hrs.

Domain of $v$ $\quad t \leq 12$ hrs $\qquad v < 100$

$\therefore 83\frac{1}{3} \leq v \leq 100$. ①

Since there is no turning point in domain then minimum cost is at the endpoint of domain ①

$v = 83\frac{1}{3} \quad C = 1349$

$v = 100 \quad C = 1430$

$\therefore$ minimum cost when $v = 83\frac{1}{3}$ km/hr. ①